ON THE NUMBER OF ISOLATED ZEROS OF PSEUDO-ABELIAN INTEGRALS: DEGENERACIES OF THE CUSPIDAL TYPE

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Abstract

We consider a multivalued function of the form $H_\varepsilon = P_\varepsilon^{\alpha_0} \prod_{i=1}^k P_i^{\alpha_i}, P_i \in \mathbb{R}[x,y], \alpha_i \in \mathbb{R}_+^*$, which is a Darboux first integral of polynomial one-form $\omega = M_\varepsilon \frac{dH_\varepsilon}{H_\varepsilon} = 0, M_\varepsilon = P_\varepsilon \prod_{i=1}^k P_i$. We assume, for $\varepsilon = 0$, that the polycycle $\{H_0 = H = 0\}$ has only cuspidal singularity which we assume at the origin and other singularities are saddles.

We consider families of Darboux first integrals unfolding H_ε (and its cuspidal point) and pseudo-Abelian integrals associated to these unfolding. Under some conditions we show the existence of uniform local bound for the number of zeros of these pseudo-Abelian integrals.

Keywords. integrable systems, blowing-up, singular foliations, singularities, abelian functions

1 Formulation of main results

In this paper, we study a non generic case. Other non generic cases have been studied in [1,3,4,5]. Pseudo-Abelian integrals appear as the linear principal part of the displacement function in polynomial perturbation of Darboux integrable case.

More precisely consider Darboux integrable system ω given by

$$\omega = M d\log H, \tag{1}$$

where

$$M = \prod_{i=0}^{k} P_i, \quad H = \prod_{i=0}^{k} P_i^{\alpha_i}, \quad \alpha_i > 0, \quad P_i \in \mathbb{R}[x,y]. \tag{2}$$

Now we consider an unfolding ω_ε of Darboux integrable system ω, where ω_ε are one-forms with first integral

$$H_\varepsilon = P_\varepsilon^{\alpha} \prod_{i=1}^{k} P_i^{\alpha_i}, \quad \omega_\varepsilon = M_\varepsilon d\log H_\varepsilon, \quad M_\varepsilon = P_\varepsilon \prod_{i=1}^{k} P_i. \tag{3}$$

where the polynomial P_0 has a cuspidal singularity at $p_0 = (0,0)$, i.e. $P_0(x,y) = y^2 - x^3 + \mathcal{O}((x,y)^4)$. For non zero ε, the polynomial $P_\varepsilon = y^2 - x^3 - \varepsilon x^2 + \mathcal{O}((x,y,\varepsilon)^4)$.

Choose a limit periodic set i.e. bounded component of $\mathbb{R}^2 \setminus \{\prod_{i=0}^k P_i = 0\}$ filled cycles $\gamma(h) \subset \{H = h\}, h \in (0,a)$. Denote by $D \subset H^{-1}(0)$ the polycycle which is in the boundary of this limit periodic set.

Consider the unfolding $\omega_\varepsilon = M_\varepsilon d\log H_\varepsilon$ of the form ω. The foliation ω_ε has a maximal nest of cycles $\gamma(\varepsilon, h) \subset \{H_\varepsilon = h\}, h \in (0, a(\varepsilon))$, filling a connected component of $\mathbb{R}^2 \setminus \{H_\varepsilon = 0\}$ whose boundary is a polycycle D_ε close to D. Assume moreover that the foliation $\omega_\varepsilon = 0$ has no singularities on $\mathrm{Int} D_\varepsilon$.

Consider pseudo-Abelian integrals of the form

$$I(\varepsilon, h) := \int_{\gamma(\varepsilon,h)} \eta_2, \quad \eta_2 = \frac{\eta_1}{M_\varepsilon} \tag{4}$$

where η_1 is a polynomial one-form of degree at most n.

This integral appears as the linear term with respect to β of the displacement function of a polynomial perturbation

$$\omega_{\varepsilon,\beta} = \omega_\varepsilon + \beta\eta_1 = 0. \tag{5}$$

We assume the following genericity assumptions

1. The level curves $P_i = 0, i = 1, \ldots, k$ are smooth and $P_i(0,0) \neq 0$.
2. The level curves $P_\varepsilon = 0, P_i = 0, i = 1, \ldots, k$, intersect transversaly two by two.

Theorem 1. *Under the genericity assumptions there exists a bound for the number of isolated zeros of the pseudo-Abelian integrals $I(\varepsilon, h) = \int_{\gamma(\varepsilon,h)} \eta_2$ in $(0, a(\varepsilon))$. The bound is locally uniform with respect to all parameters in particular in ε.*

Let $\mathcal{F}_1 : \{\omega_\varepsilon = 0\}, \mathcal{F}_2 : \{d\varepsilon = 0\}$ are the foliations of dimension two in complex space of dimension three with coordinates (x, y, ε).

Let $\mathcal{F}$ be the foliation of dimension one on the complex space of dimension three with coordinates (x, y, ε) which is given by the intersection of leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ (i.e. given by the 2-form $\Omega = \omega_\varepsilon \wedge d\varepsilon$). This foliation has a cuspidal singularity at the origin (a cusp).

We want to study the analytical properties of the foliation $\mathcal{F}$ in a neighborhood of the cusp. For this reason we make a global blowing-up of the cusp of the product space (x, y, ε) of phase and parameter spaces. We want our blow-up to seperate the two branche of the cusp. This requirements leads to the quasi-homogeneous blowing-up of weight $(2, 3, 2)$.

Remark 1. *In term of first integrals, the foliation $\mathcal{F}$ is given by two first integrals*

$$H(x, y, \varepsilon) = h, \quad \varepsilon = s.$$

2 Quasi-homogeneous blowing-up of $\mathcal{F}$

Recall the construction of the quasi-homogenous blowing-up. We define the weighted projective space $\mathbb{CP}^2_{2:3:2}$ as factor space of $\mathbb{C}^3$ by the $\mathbb{C}^*$ action $(x, y, \varepsilon) \mapsto (t^2x, t^3y, t^2\varepsilon)$. The quasi-homogeneous blowing-up of $\mathbb{C}^3$ at the origin is defined as the incidence three dimensional manifold $W = \{(p, q) \in \mathbb{CP}^2_{2:3:2} \times \mathbb{C}^3 : \exists t \in \mathbb{C} : (q_1, q_2, q_3) = (t^2p_1, t^3p_2, t^2p_3)\}$, where $(q_1, q_2, q_3) \in \mathbb{C}$ and $[(p_1, p_2, p_3)] \in \mathbb{CP}^2_{2:3:2}$.

The quasi-homogeneous blowing-up $\sigma : W \to \mathbb{C}^3$ is just the restriction to W of the projection $\mathbb{CP}^2_{2:3:2} \times \mathbb{C}^3$.

We will need explicit formula for the blow-up in the standard affine charts of W. The projective space $\mathbb{CP}^2_{2:3:2}$ is covred by three affine charts: $U_1 = \{x \neq 0\}$ with coordinates (y_1, z_1), $U_2 = \{y \neq 0\}$ with coordintaes (x_2, z_2) and $U_3 = \{\varepsilon \neq 0\}$ with coordinates (x_3, y_3).

The transition formula follow from the requirement that the points $(1, y_1, z_1), (x_2, 1, z_2)$ and $(x_3, y_3, 1)$ lie on the same orbit of the action:

$$F_2 : (y_1, z_1) \mapsto \left(x_2 = 1/y_1^{2/3}, z_2 = z_1/y_1\sqrt{y_1}\right)$$
$$F_3 : (y_1, z_1) \mapsto (x_3 = 1/z_1, y_3 = y_1/z_1\sqrt{z_1}).$$

These affine charts define affine charts on W, with coordinates (y_1, z_1, t_1), (x_2, z_2, t_2) and (x_3, y_3, t_3). The blow-up σ is written as

$$\sigma_1 : \quad x = t_1^2, \qquad y = t_1^3y_1, \qquad \varepsilon = t_1^2z_1 \tag{6}$$
$$\sigma_2 : \quad x = t_2^2x_2, \qquad y = t_2^3, \qquad \varepsilon = t_2^2z_2 \tag{7}$$
$$\sigma_3 : \quad x = t_3^2x_3, \qquad y = t_3^3y_3, \qquad \varepsilon = t_3^2. \tag{8}$$

We apply this blow-up σ to the one-dimensional foliation $\mathcal{F}$. Let $\sigma^{-1}\mathcal{F}$ the lifting of the foliation $\mathcal{F}$ to the complement This foliation has a cuspidal singularity at the origin. The pull-back foliation $\sigma^*\mathcal{F}$ will be called the strict transform of the foliation $\mathcal{F}$ is defined by the pull-back $\sigma^*\Omega = \sigma^*\ (\omega_\varepsilon \wedge d\varepsilon)$ divided by a suitable power of the function defining the exceptional divisor. In this charts $U_j, j = 1, 2, 3$ we have

$$\sigma_1^*\Omega = x^2\Omega_1, \quad \sigma_2^*\Omega = y^3\Omega_2, \quad \sigma_3^*\Omega = \varepsilon^2\Omega_3,$$

where

$$\Omega_1 = (6y_1^2 - 6 - 4z_1)dx \wedge dz_1 + 4y_1z_1dy_1 \wedge dx + 2xy_1dy_1 \wedge dz_1, \tag{9}$$

$$\Omega_2 = (6 - 6x_2^3 - 4x_2^2z_2)dy \wedge dz_2 + (-6z_2x_2^2 - 4x_2z_2^2)dx_2 \wedge dy \tag{10}$$

$$+ (-3yx_2^2 - 2yx_2z_2)dx_2 \wedge dz_2, \tag{11}$$

$$\Omega_3 = (-6x_3^2 - 4x_3)dx_3 \wedge d\varepsilon + 4y_3dy_3 \wedge d\varepsilon. \tag{12}$$

Remark 2. *In term of first integrals, the foliation $\sigma^*\mathcal{F}$ is given by two first integrals*

$$\sigma^*H(x, y, \varepsilon) = h, \quad \sigma^*\varepsilon = s,$$

In particular in a neighborhood of the exceptional divisor the restrictions of the foliation $\sigma^*\mathcal{F}$ to the charts U_1 and U_3 are given respectively, by

$$\psi_1 = H(t_1^2, t_1^3y_1, t_1^2z_1) = x^3(y_1^2 - 1) = h, \quad \varphi_1 = xz_1 = s, \tag{13}$$

$$\psi_3 = H(t_3^2x_3, t_3^3y_3, t_3^2) = \varepsilon^3(y_3^2 - x_3^2 - x_3^3) = h, \quad \varphi_3 = \varepsilon = s, \tag{14}$$

where $\{x = 0\}$ and $\{\varepsilon = 0\}$ are local equations of the exceptional divisor respectively.

3 Singular locus of the foliation $\sigma^*\mathcal{F}$

In this section, we compute the singular locus of the pull-back $\sigma^*\Omega$ in a neighborhood of the exceptional divisor $\mathbb{CP}^2_{2:3:2}$. We check it in each chart seperatly.

In the chart U_1, the zeros locus of the form Ω_1 in a neighborhood of the exceptional divisor $\{x = 0\}$ consists of germs of two curves $\{y_1 = \pm 1, z_1 = 0\}$ and a two singular points $p_1 = (0, 1, 0), p_2 = (0, -1, 0)$ generated by the quasi-homogeneous blowing-up.

In the chart U_3, the zeros locus of the form Ω_3 in a neighborhood of the exceptional divisor $\{\varepsilon = 0\}$ consists of $p_3 = (0, 0, 0)$ (Morse point) and $p_4 = (-\frac{2}{3}, 0, 0)$ (center). The singularities of this foliation are the line of Morse points $x_3 = 0, y_3 = 0$, the lines of centers $x_3 = -\frac{2}{3}, y_3 = 0$ and the transform strict of $\{y^2 - x^3 - x^2\varepsilon = 0\}$.

Proposition 1. *The singularities of $\sigma^*\mathcal{F}$ are located at the points p_1, p_2, p_3 and p_4. The points p_1, p_2 and p_3 are linearisable saddles and the point p_4 is a center.*

Proof. Since $\sigma : W \to \mathbb{C}^3$ is a biholomorphism autside the exceptional divisor $\mathbb{CP}^2_{2:3:2}$, all singularities of $\sigma^*\mathcal{F}$ on $\mathbb{C}^3 \setminus \{x = 0\}$ correspond to singularities of $\mathcal{F}$. Thus, it suffices to compute the singularities of $\sigma^*\mathcal{F}$ on the exceptional divisor $\{x = 0\}$. More precisely, we consider the foliation on neighborhood of $\mathbb{CP}^2_{(2:3:2)}$ (the exceptional divisor) generated by the blown-up one-form $\sigma^*\Omega$. Let ψ_1, ψ_3 are the functions given in (13) and (14).

(1) In the chart U_1, near the divisor exceptional and for $|z_1| \leq \epsilon$ for ϵ sufficiently small, the foliation $\sigma^*\mathcal{F}$ is given by two first integrals

$$G_1 = \varphi_1^3\psi_1^{-1} = z_1^3(y_1^2 - (1 + z_1))^{-1}V^{-1} = s^3h^{-1}, \quad \varphi_1 = xz_1 = s.$$

where V is analytic function such that $V(0,0,0) \neq 0$. In particular on the exceptional divisor $\{x=0\}$ the foliation $\sigma^*\mathcal{F}$ is given by the levels $G_1 = s^3h^{-1} = t$.

Now we calculate the eigenvalues at p_1 and p_2. The vector field V_1 generating the foliation $\sigma^*\mathcal{F}$ is given by

$$V_1(x, y_1, z_1) = \beta_1 x\frac{\partial}{\partial x} + \beta_2 y_1\frac{\partial}{\partial y_1} + \beta_3 z_1\frac{\partial}{\partial z_1},$$

where the vector $(\beta_1, \beta_2, \beta_3)$ satisfies the following equations

$$< (\beta_1, \beta_2, \beta_3), (3, 1, 0) >= 0, < (\beta_1, \beta_2, \beta_3), (1, 0, 1) >= 0$$

here $<,>$ be the usual scalar product on $\mathbb{C}^3$. By simple computation, we obtain $\beta_1 = 1, \beta_2 = -3$ and $\beta_3 = -1$.

(2) In the chart U_3, near the exceptional divisor $\{\varepsilon = 0\}$, the foliation $\sigma^*\mathcal{F}$ is given by

$$G_3 = \varphi_3^3\psi_3^{-1} = (y_3^2 - x_3^2(1+x_3))^{-1} = s^3h^{-1}, \quad \varphi_3 = \varepsilon = s.$$

In particular the restriction of this foliation to the exceptional divisor $\{\varepsilon = 0\}$, by Morse lemma we can put the function $1/G_3$ to the normal form $y_3^2 - z_3^2$ in a neighborhood of p_3 (we put the variable change $z_3 = \pm x_3(1+x_3)^{1/2}$). On other hand the Hessian matrix of $1/G_3$ at the point p_4 has two positive eigenvalus. $\square$

4 The different scaled variations of $\delta(s,t)$

In this section, we compute the scaled variations with respect to differents variables s and t of the integrals of the blown- up one form $\sigma_1^*\eta_2$ along the different relatives cycles using the same technics of [5].

Proposition 2. *The computation of the different scaled variations of the cycle $\delta(s,t)$ us gives*

1. *For $t \in [0, 2N]$, the cycle $\delta(s,t)$ satisfies a iterated scaled variations with respect to t of the form*

$$\mathcal{V}ar_{(t,3)} \circ \mathcal{V}ar_{(t,-1)} \circ \mathcal{V}ar_{(t,-\alpha_1)} \circ \ldots \circ \mathcal{V}ar_{(t,-\alpha_k)}\delta(s,t) = 0. \tag{15}$$

2. *For $t \in [N, +\infty]$, the cycle $\delta(s,t)$ satisfies a iterated scaled variations with respect to $1/t$ of the form*

$$\mathcal{V}ar_{(1/t,-3)} \circ \mathcal{V}ar_{(1/t,1)} \circ \mathcal{V}ar_{(1/t,1)} \circ \mathcal{V}ar_{(1/t,\alpha_1)} \circ \ldots \circ \mathcal{V}ar_{(1/t,\alpha_k)}\delta(s,1/t) = 0. \tag{16}$$

3. *Near $s = 0$, we have*

$$\mathcal{V}ar_{(s,1)} \circ \mathcal{V}ar_{(s,1)}\delta(s,t) = \mathcal{V}ar_{(s,1)}(\tilde{\delta}(s,t)) = 0, \tag{17}$$

where $\mathcal{V}ar_{(s,1)}\delta(s,t) = \tilde{\delta}(s,t)$ is a figure eight cycle.

Proof. As in [5], there exist a some local chart with coordinates (u,v,w) defined in a some neighborhood of each separatrix of polycycle such that the foliation $\sigma^*\mathcal{F}$ is defined by two first integrals. Precisely:

1. for $t \in [0, 2N]$, there exist a local chart $(V_{div}, (u,v,w))$ defined in neighborhood of the separatrix δ_{div} such the foliation $\sigma^*\mathcal{F}$ by two first integrals

$$F_1 = w^3(v-1)^{-1}(v+1)^{-1} = t, \quad F_2 = uw = s,$$

2. for $t \in [N, +\infty]$, there exists a local chart $(V_{div}^{+}, (u, v, w))$ defined in neighborhood of the separatrix δ_{div}^{+} such that the foliation $\sigma_1^*\mathcal{F}$ is defined by two first integrals

$$F_1 = w^3(v+2)^{-1}v^{-1} = t, \quad F_2 = uw = s,$$

3. for $t \in [N, +\infty]$, there exists a local chart $(V_{div}^{-}, (u, v, w))$ defined in a neighborhood of the separatrix δ_{div}^{-} such that the foliation $\sigma_1^*\mathcal{F}$ is defined by two first integrals

$$F_1 = w^3(v-2)^{-1}v^{-1} = t, \quad F_2 = uw = s.$$

In second step we prove that each relative cycle can be chosen as a lift of a path contained in the separatrix associated to this relative cycle. Precisely:

1. on the chart $(V_{div}, (u, v, w))$, the linear projection $\pi(u, v, w) = v$ is every where transverse to the levels of the foliation $\sigma^*\mathcal{F}$ which corresponds simply to the graphs of the multivalued functions

$$v \mapsto (u, w) = \left(st^{-\frac{1}{3}}(v-1)^{-\frac{1}{3}}(v+1)^{-\frac{1}{3}}, t^{\frac{1}{3}}(v-1)^{\frac{1}{3}}(v+1)^{\frac{1}{3}}\right),$$

2. on the chart $(V_{div}^{+}, (u, v, w))$, the linear projection $\pi(u, v, w) = v$ is every where transverse to the levels of the foliation $\sigma^*\mathcal{F}$ which corresponds simply to the graphs of the multivalued functions

$$v \mapsto (u, w) = \left(st^{-\frac{1}{3}}v^{-\frac{1}{3}}(v+2)^{-\frac{1}{3}}, t^{\frac{1}{3}}v^{\frac{1}{3}}(v+2)^{\frac{1}{3}}\right),$$

3. on the chart $(V_{div}^{-}, (u, v, w))$, the linear projection $\pi(u, v, w) = v$ is every where transverse to the levels of the foliation $\sigma^*\mathcal{F}$ which corresponds simply to the graphs of the multivalued functions

$$v \mapsto (u, w) = \left(st^{-\frac{1}{3}}v^{-\frac{1}{3}}(v-2)^{-\frac{1}{3}}, t^{\frac{1}{3}}v^{\frac{1}{3}}(v-2)^{\frac{1}{3}}\right).$$

In third step, we compute the different scaled variations of relatives cycles using the local expression of two first integrals F_1 and F_2 above near the singular points p_1, p_2 and p_3. Recall that the scaled variation of a relative cycle $\delta(s)$ is given by

$$\mathcal{V}ar_{(s,\beta)}\delta(s) = \delta(se^{i\pi\beta}) - \delta(se^{-i\pi\beta}).$$

In the local chart $(V_{div}^{+}, (u, v, w))$, the restriction of the blown-up foliation $\sigma_1^*\mathcal{F}$ to the transversals sections $\Sigma_{div}^{-} = \{w = 1\}$ (near the point p_3) and $\Omega_{+} = \{u = 1\}$ (near the point p_1) is given respectively by

$$F_1|_{\Sigma_{div}^{-}} = \frac{1}{v} = t, \quad F_2|_{\Sigma_{div}^{-}} = u = s,$$
$$F_1|_{\Omega_{+}} = \frac{w^3}{v} = t, \quad F_2|_{\Omega_{+}} = w = s.$$

Let us fix $t \in [N, +\infty]$. We observe that the restriction of the foliation $\sigma_1^*\mathcal{F}$ to the transversal section $\Sigma_{div}^{+} = \{w = 1\}$ is analytic with respect to s. Then, after taking an scaled variation with respect to s, the relative cycle $\delta_{div}^{+}(s, t)$ is replaced by a loop θ_1, modulo homotopy, which consists of line segment $\ell_{31} = [p_3, p_1]$ connecting the Morse point p_3 with the point p_1 encircling the latter along a small counterclockwise circular arc α_1 and then returning along the segment $\ell_{13} = [p_1, p_3]$. The loop θ_1 can be moved along the complex curve $\{u = w = 0\}$. Then, we have

$$\mathcal{V}ar_{(s,1)}\delta_{div}^{+}(s, t) = \theta_1 = \ell_{31}\alpha_1\ell_{13}.$$

The same computation of the scaled variation with respect to s for the relative cycle $\delta_{div}^{-}(s, t)$ gives us a loop θ_3, modulo homtopy, which can be moved along the complex plane $\{u = w = 0\}$. The loop θ_3

consists of line segment $\ell_{32} = [p_3, p_2]$ connecting the point p_3 with the point p_2 encircling the latter along a small counterclockwise circular arc α_3 and then returning along the segment $\ell_{23} = [p_2, p_3]$. Then, we have

$$\mathcal{V}ar_{(s,1)}\delta^{-}_{div}(s,t) = \theta_3 = \ell_{32}\alpha_3\ell_{23}.$$

In the local chart $(V_{div}, (u, v, w))$, we define the transversal section $\Omega_+ = \{u = 1\}$ (resp $\Omega_+ = \{u = 1\}$) near p_1 (resp near p_2). The restriction of the foliation $\sigma_1^*\mathcal{F}$ to the transversal section Ω_+ is given by

$$F_1|_{\Omega_+} = \frac{w^3}{v} = t, \quad F_2|_{\Omega_+} = w = s.$$

On the second step let us fix $t \in [0, 2N]$. After taking an scaled variation with respect to s, the relative cycle $\delta_{div}(s,t)$ is replaced by a figure eight cycle which can be moved along the complex line $C^t_{div} = \{x = 0, G_1 = t\}$ of the foliation $\sigma_1^*\mathcal{F}$. This case is similar to the classical situation which is studied by Bobieński and Mardešić in [2].

Now using the analycity of the lifting $\sigma^{-1}\mathcal{F}$ with respect to s, the scaled variation of the cycle of integration $\delta(s,t)$ with respect to s is equal to the scaled variation with respect to s of the following difference $\delta^{+}_{div}(s,t) - \delta^{-}_{div}(s,t)$ which is equal, modulo homotopy, to the cycle $\theta_1\theta_3^{-1}$, where θ_3^{-1} is the inverse of the loop θ_3. Shematically, the loop $\theta_1\theta_3^{-1}$ is a figure eight cycle. $\square$

Remark 3.

- In the local chart $(V^{+}_{div}, (u, v, w))$ (resp $V^{-}_{div}, (u, v, w))$, the loop θ_1 (resp θ_3) generating the fundamental group of the complex plane $\{u = w = 0\} \setminus \{p_1\}$ (resp $\{u = w = 0\} \setminus \{p_2\}$) with base point p_3.

- By the univalness of the blown-up one form $\sigma_1^*\eta_2$, we have

$$\mathcal{V}ar_{(t,\alpha)} \int_{\delta(s,t)} \sigma_1^*\eta_2 = \int_{\mathcal{V}ar_{(t,\alpha)}\delta(s,t)} \sigma_1^*\eta_2.$$

5 Proof of the Theorem

In this section we first take benefit from the blowing-up in the family to prove our principal theorem. the proof is analoguous of the following :

Theorem 2. *There exists a bound of the number of zeros of the function $t \mapsto J(s,t)$, for $t \in [0, +\infty]$ and $s > 0$ sufficiently small. This bound is locally with respect to all parameters uniform, in particular with respect to s.*

Let $\beta = (\beta_1, \ldots, \beta_{k+2})$ where $\beta_1 = 3, \beta_2 = -1, \beta_3 = -\alpha_1, \ldots, \beta_{k+2} = -\alpha_k$. Let D_1 is slit annulus in the complex plane $\mathbb{C}^*_t$ with boundary ∂D_1. This boundary is decomposed as follows $\partial D_1 = C_{R_1} \cup C_{r_1} \cup C^{\pm}$, where $C_{R_1} = \{|t| = R_1, |\arg t| \leq \alpha\pi\}, C^{\pm} = \{r_1 < |t| < R_1, |\arg t| = \pm\alpha\}$ and $C_{r_1} = \{|t| = r_1, |\arg t| \leq \alpha\pi\}$.

Petrov's method gives us that the number of zeros $\#Z(J(s,t))$ of the function $J(s,t)$ in slit annulus D_1 is bounded by the increment of the argument of $J(s,t)$ along ∂D_1 divided by 2π i.e.

$$\begin{aligned}\#Z(J(s,t)|_{D_1}) \leq & \frac{1}{2\pi}\Delta\arg(J(s,t)|_{\partial D_1}) = \frac{1}{2\pi}\Delta\arg(J(s,t)|_{C_{R_1}}) \\ & + \frac{1}{2\pi}\Delta\arg(J(s,t)|_{C^{\pm}}) + \frac{1}{2\pi}\Delta\arg(J(s,t)|_{C_{r_1}})\end{aligned}$$

(**A**) The increment of argument $\Delta\arg(J(s,t)|_{C_{R_1}})$ is uniformly bounded by Gabrielov's theorem [6].

(**B**) We use the Schwartz's principle

$$Im(J(s,t))|_{C^{\pm}} = \mp 2i\mathcal{V}ar_{(t,\alpha)}J(s,t).$$

Thus, the increments of argument along segments $C^{\pm}$ are bounded by zeros of the variation $\mathcal{V}ar_{(t,\alpha)}J(s,t)$ on segment (r,R). By identity (18), the function $\mathcal{V}ar_{(t,\beta_i)}J(s,t)$ can be written as follows

$$\begin{aligned}\mathcal{V}ar_{(t,\beta_i)}J(s,t) &= K(t^{\frac{\beta_1}{\beta_i}},\dots,t^{\frac{\beta_{k+\mu}}{\beta_i}},s;\log s)\\ &= K(e^{\frac{\beta_1}{\beta_i}\log t},\dots,e^{\frac{\beta_{k+\mu}}{\beta_i}\log t},e^{\log s};\log s)\end{aligned}$$

where K is a meromorphic function. The function $\mathcal{V}ar_{(t,\beta_i)}J(s,t)$ is logarithmico-analytic function of type 1 in the variable s (see [9]). Then, there exist a finit recover of $\mathbb{R}^{k+\mu+1}\times\mathbb{R}$ by a logarithmico-exponential cylinders, using Rolin-Lion's theorem [9], such that on each cylinder of this family we have

$$\mathcal{V}ar_{(t,\beta_i)}J(s,t) = y_0^{r_0}y_1^{r_1}A(t)U(t,y_0,y_1),$$

with $y_0 = s-\theta_0(t), y_1 = \log y_0 - \theta_1(t)$, where θ_0,θ_1,A are logarithmico-exponential functions and U is a logarithmico-exponential unity function. As the number of zeros of a logarithmico-exponential function is bounded, the number of zeros of $\mathcal{V}ar_{(t,\beta_i)}J(s,t)$ is bounded.

(**C**) Finally, we estimate the increment of argument of J along the small arc C_{r_1}. Then, it is necessarily to study the increment of argument of the leading term of the function J at $t=0$.

Lemma 1. *The increment of the argument of $J(s,t)$ along the small circular arc C_{r_1} can be estimated by the increment of the argument of a some meromorphic function $F(s,t)$.*

Proof. The problem of the estimation of the increment of the argument of $J(s,t)$ along the circular arc C_{r_1} consist that the principal part of the function J contains the term $\log s \to -\infty$ as $s\to 0$. To resolve this problem we make a blowing-up at the origin in the total space with coordinates (x,y,z) where

$$x = J_1(s,t),\quad y = J_2(s,t),\quad z = (\log s)^{-1}.$$

The function $J(s,t)$ can be rewritten as follows

$$J(s,t) = J_1(s,t) + J_2(s,t)\log s = ((\log s)^{-1}J_1(s,t) + J_2(s,t))\log s = (zx+y)z^{-1}.$$

Thus, for $z^{-1}\in\mathbb{R}$ be sufficiently small, we have

$$\arg(J(s,t)) = \arg((zx+y)z^{-1}) = \arg(zx+y).$$

To estimate the increment of argument of $zx+y$ uniformly with respect to $s>0$ we make a quasi-homogeneous blowing-up π_1 with weight $(\frac{1}{2},1,\frac{1}{2})$ of the polynomial $zx+y$ at $C_1=\{x=y=z=0\}$ (the centre of blowing-up). The explicit formula of the quasi-homogeneous blowing-up π_1 in the affine charts $T_1=\{x\neq 0\}, T_2=\{y\neq 0\}$ and $T_3=\{z\neq 0\}$ is written respectively as

$$\begin{aligned}\pi_{11}&: x=\sqrt{x_1},\quad y=y_1x_1,\quad z=z_1\sqrt{x_1},\\ \pi_{12}&: x=x_2\sqrt{y_2},\quad y=y_2,\quad z=z_2\sqrt{y_2},\\ \pi_{13}&: x=x_3\sqrt{z_3},\quad y=y_3z_3,\quad z=\sqrt{z_3}.\end{aligned}$$

The pull-back $\pi_1^*(zx+y)$ is given, in different charts, by

$$\begin{aligned}\pi_{11}^*(zx+y) &= x_1(z_1+y_1) = d_1P_1(x_1,y_1,z_1),\\ \pi_{12}^*(zx+y) &= y_2(x_2z_2+1) = d_2P_2(x_2,y_2,z_2),\\ \pi_{13}^*(zx+y) &= z_3(x_3+y_3) = d_3P_3(x_3,y_3,z_3).\end{aligned}$$

where $d_i = 0$ and $P_i = 0$ are equations of exceptional divisor and the strict transform of $zx + y = 0$ respectively.

Observe that $P_i = 0, i = 1, 3$ has not a normal crossing with tha exceptional divisor $d_i = 0, i = 1, 3$. To resolve this problem we make a second blowing-up π_2 with centre a subvariety C_2 which is given, in differents charts, as following:

1. In the chart T_1, choose a local coordinate chart with coordinates (x_1, y_1, z_1) in which $C_2 = \{y_1 = z_1 = 0\}$. Then $\pi_2^{-1}(C_2)$ is covred by two coordinates charts U_{y_1} and U_{z_1} with coordinate $(\tilde{x}_1, \tilde{y}_1, \tilde{z}_1)$ where in y_1-chart U_{y_1} the blowing-up π_2 is given by $x_1 = \tilde{x}_1, y_1 = \tilde{y}_1, z_1 = \tilde{z}_1\tilde{y}_1$ and in z_1-chart U_{z_1} the blowing-up π_2 is given by $x_1 = \tilde{x}_1, y_1 = \tilde{y}_1\tilde{z}_1, z_1 = \tilde{z}_1$.

2. In the chart T_2, the blowing-up π_2 is a biholomorphism (π_2 is a proper map).

3. In this chart T_3, choose a local coordinate chart with coordinates (x_3, y_3, z_3) in which $C_2 = \{x_3 = y_3 = 0\}$. Then $\pi_2^{-1}(C_2)$ is covred by two coordinates charts U_{x_3} and U_{y_3} with coordinate $(\tilde{x}_3, \tilde{y}_3, \tilde{z}_3)$ where in x_3-chart U_{x_3} the blowing-up π_2 is given by $x_3 = \tilde{x}_3, y_3 = \tilde{y}_3\tilde{x}_3, z_3 = \tilde{z}_3$ and in y_3-chart U_{y_3} the blowing-up π_2 is given by $x_3 = \tilde{x}_3\tilde{y}_3, y_3 = \tilde{y}_3, z_3 = \tilde{z}_3$.

The pull-back $\pi_1^*(zx + y)$ is given, in different charts, by

- In the y_1-chart U_{y_1}, the transformation of the pull-back $\pi_1^*(zx+y)$ by the blowing-up π_2 is given by
$$\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_1 P_1(x_1, y_1, z_1)) = \tilde{x}_1\tilde{y}_1(\tilde{z}_1 + 1) \overset{0}{\approx} \tilde{x}_1\tilde{y}_1 = J_2(s, t) = F(s, t).$$

- In the z_1-chart U_{z_1}, the transformation of the pull-back $\pi_1^*(zx+y)$ by the blowing-up π_2 is given by
$$\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_1 P_1(x_1, y_1, z_1)) = \tilde{z}_1\tilde{x}_1(\tilde{y}_1 + 1) \overset{0}{\approx} \tilde{x}_1\tilde{z}_1 = (\log s)^{-1} J_1(s, t) = F(s, t).$$

- In the chart T_2, we have
$$\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_2 P_2(x_2, y_2, z_2)) = d_2 P_2(x_2, y_2, z_2) = (\log s)^{-1} J_1(s, t) + J_2(s, t) = F(s, t).$$

- In the x_3-chart U_{x_3}, the transformation of the pull-back $\pi_1^*(zx+y)$ by the blowing-up π_2 is given by
$$\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_3 P_3(x_3, y_3, z_3)) = \tilde{z}_3\tilde{x}_3(\tilde{y}_3 + 1) \overset{0}{\approx} \tilde{x}_3\tilde{z}_3 = (\log s)^{-1} J_1(s, t) = F(s, t).$$

- In the y_3-chart U_{y_3}, the transformation of the pull-back $\pi_1^*(zx+y)$ by the blowing-up π_2 is given by
$$\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_3 P_3(x_3, y_3, z_3)) = \tilde{z}_3\tilde{y}_3(\tilde{x}_3 + 1) \overset{0}{\approx} \tilde{y}_3\tilde{z}_3 = J_2(s, t) = F(s, t).$$

Finally, we distinguish three cases:

1. $\arg_{C_{r_1}} J(s, t) = \arg_{C_{r_1}} ((\log s)^{-1} J_1(s, t)) = \arg_{C_{r_1}} J_1(s, t)$, $((\log s)^{-1} \in \mathbb{R})$
2. $\arg_{C_{r_1}} J(s, t) = \arg_{C_{r_1}} J_2(s, t)$,
3. In the chart T_2, the function $F(s, t) = ((\log s)^{-1} J_1(s, t)) + J_2(s, t)$ is meromorphic.

□

Now we define the functional space $\mathcal{P}_\beta$ which are formed of coefficients of the polynomials P_i of the Darboux first integral H, the coefficients of the polynomials R, S of the perturbative one forme η, exponents α_i and degrees $n_i = \deg P_i, n = \max(\deg R, \deg S)$. Consider the following finite dimensional functional space $\mathcal{P}_\beta$

$$\begin{aligned}\mathcal{P}_\beta(m_\beta, M_\beta; \beta_1, \ldots, \beta_{k+2}) = \{ & \sum_{j=1}^{k+2} \sum_{n,\ell} A_{j\ell n}(s) t^{\beta_j n} s^m \log^\ell(t) : \\ & A_{j\ell n}(s) \in \mathbb{C}, m_\beta < A_{j\ell n} < M_\beta, 0 \leq \ell \leq k+1 \}.\end{aligned}$$

For the first two cases, the function $J_i(s,t), i = 1, 2$ satisfies the following iterated variations equation with respect to t

$$\mathcal{V}ar_{(t,\beta_1)} \circ \ldots \circ \mathcal{V}ar_{(t,\beta_{k+2})} J_i(s,t) = 0.$$

Thus, by Lemma 4.8 from [2], there exists a non zero leading term $P_{i\beta} \in \mathcal{P}_\beta$ of $J_i(s,t)$, i=1,2 at $t = 0$ such that $|J_i(s,t) - P_{i\beta}(s,t)| = O(t^{\mu_1}), \mu_1 > 0$, uniformly in s. Moreover, the function $J_i(s,t), i = 1, 2$ satisfies the iterated variation equation

$$\mathcal{V}ar_{(s,1)} J_i(s,t) = 0.$$

Thus, we have $J_i(s,t) = O(s^{\mu_2}), \mu_2 > 0$, uniformly in t.

For each element in the parameter space, we can choose the leading term of $P_{i\beta}$. The increment of argument of this leading term is bounded by a constant $C(M_\beta, k+2, \beta_{k+2})$. Since the leading term of $P_{i\beta}$ is also the leading term of $J_i(s,t)$, the limit $\lim_{r_1 \to 0} \Delta \arg(J_i(s,t)|C_{r_1}) \leq C(M_\beta, k+2, \beta_{k+2})$.

In the chart T_2, the function F is meromorphic. Thus, this function can be rewritten as following

$$F(s,t) = (\log s)^{-1} J_1(s,t) + J_2(s,t) = G(t^{\beta_1}, \ldots, t^{\beta_k}, s, (\log s)^{-1})$$

where G is meromorphic function. The number $\#Z(G)$ of zeros of the function G is uniformly bounded. The latter claim is a direct application of fewnomials theory of Khovanskii [8]: since the functions $\epsilon_i(t) = t^{\beta_i}, \epsilon(s) = (\log s)^{-1}$ are Pfaffian functions (solutions of Pfaffian equations $t d\epsilon_i - \beta_i \epsilon_i dt = 0$ and $s d\epsilon + \epsilon^2 ds$, respectively), the upper bound for this number of zeros can be given, using Rolle-Khovanskii arguments of [7], in terms of the number of zeros of some polynomial and its derivatives. The latter are uniformly bounded by Gabrielov's theorem [6].

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